JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS **160**, 525–555 (1991)

On the Existence of Global Weak Solutions for Vlasov–Poisson–Fokker–Planck Systems*

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Submitted by Avner Friedman

Received August 28, 1989

This work studies the question of global existence of weak solutions to Vlasov–Poisson–Fokker–Planck equations with spatial, or momentum, dimensions greater than, or equal to, three. We define a concept of weak solutions to these equations and show (i) global existence for plasma physical models with spatial (or momentum) dimensions greater than or equal to three, and (ii) global existence in three spatial dimensions with arbitrary data for the stellar dynamical case. © 1991 Academic Press, Inc.

I. INTRODUCTION

The mathematical description of the state of a stellar system or a rarefied plasma has been based on collisionless models. These kinetic models are the Liouville–Newton or the Vlasov–Poisson system of equations in case the induced magnetic fields vary slowly. Such models consist of a nonlinear hyperbolic conservation law based on the underlying physics coupled with Poisson's equation for determining the self-consistent gravitational or electrostatic forces.

The model of collisionless plasmas–especially in the applied contexts of controlled fusion, and of laser fusion—is a highly idealized one. A way to incorporate collisional effects of a plasma with the background material (e.g., a plasma system in a thermal bath or "reservoir") is to model the motion of an individual particle as Brownian motion caused by collisions

* This research was supported in the initial phases by a grant from the Alexander von Humboldt Foundation for the summer of 1987 at the Ludwig-Maximilians Universität, München. The final stages of this work was supported by the National Science Foundation Grant D.M.S. 87-10292.

0022-247X/91 $3.00

with the background. This conceptual model is then analogous to the classical description of the irregular—or Brownian—motion exhibited by particles of colloidal size immersed in a fluid. In the stellar dynamical context, one of the fundamental problems is to incorporate in the framework of a general theory the effect of encounters between stars; and stellar encounters under Newtonian inverse square attractions influence the motions of stars in the manner of Brownian motion [3], pp. 385–386].

The system of mathematical equations used to examine the resulting particle motion is the system of *Langevin equations*, in which a suitable stochasticity is included: In order to take into account the complex interaction between the particles and the background [2], the relevant system of stochastic differential equations is

$$\begin{aligned} dx &= v\,dt, \\ dv &= (\mathbf{E}(t, x) - \beta v)\,dt + \sqrt{2\sigma}\,d\mathbf{b}, \end{aligned}$$

where $\mathbf{E}(t, x)$ denotes the (self-consistent) electrostatic or gravitational field; β is a viscosity parameter; and $\sigma = \beta \mathscr{K} T/\mathfrak{m}^-$. Here, T denotes the absolute temperature of the medium in which the plasma is confined, and $\mathscr{K}$ is the Maxwell–Boltzmann constant. The parameter $\mathfrak{m}^-$ denotes the mass of an electron and $\mathbf{b}$ is the standard N-dimensional Brownian motion.

The Vlasov–Poisson–Fokker–Planck equations result when one incorporates the above Langevin system of equations into the Vlasov or Liouville equation for determining the dynamic behavior of the expected distribution of particles with respect to position and momentum. Letting $f(t, x, v)$ denote this distribution at time t, we see that our Vlasov–Poisson–Fokker–Planck system is [2]

$$\begin{aligned} &\frac{\partial f}{\partial t}(t, x, v) + v \cdot \nabla_x f(t, x, v) + (\mathbf{E}(t, x) - \beta v) \cdot \nabla_v f(t, x, v) \\ &\qquad = N\beta f(t, x, v) + \sigma \Delta_v f(t, x, v), \; (t, x, v) \in [0, \infty) \times \mathbb{R}^N \times \mathbb{R}^N, \end{aligned} \tag{1.1}$$

where

$$\begin{aligned} \mathbf{E}(t, x) &:= -\frac{\lambda}{\omega_N} \int_{\mathbb{R}^N} \frac{x - y}{|x - y|^N} \int_{\mathbb{R}^N} [f(t, y, v) - N_0(y, v)]\,dv\,dy, \\ &= -\nabla_x \mathbf{U}(t, x), \end{aligned} \tag{1.2}$$

with

$$\mathbf{U}(t, x) := -\frac{\lambda}{(N-2)\,\omega_N} \int_{\mathbb{R}^N} |x - y|^{-(N-2)} \int_{\mathbb{R}^N} [f(t, y, v) - N_0(y, v)]\,dv\,dy, \tag{1.3}$$

and

$$f|_{t=0} = f_0(x, v), \qquad (x, v) \in \mathbb{R}^N \times \mathbb{R}^N. \tag{1.4}$$

Here, $\mathbb{R}^N$ denotes real N-dimensional positional or momentum space. In the plasma physical case, $\lambda := -4\pi\mathfrak{e}^2/\mathfrak{m}^-$, where $\mathfrak{e}$ is a unit of electric charge. The electrostatic or gravitational forces are denoted by $\mathbf{E}(t, x)$, with $\mathrm{U}(t, x)$, the potential. The function $N_0(x, v)$ represents the distribution of ions, which by their inertia form a uniform neutralizing background. In the stellar dynamical case, $\lambda := 4\pi\mathfrak{g}\mathfrak{m}$, where $\mathfrak{g}$ is the gravitational constant and $\mathfrak{m}$ is the mass of a stellar particle. A discussion of the parameters β and σ can be found in [2, 3].

The existence of classical solutions to Vlasov–Poisson–Fokker–Planck models was studied in [4, 13]. In [4], P. Degond examined a type of Vlasov–Fokker–Planck model, which had a diffusion term without dynamical friction. He employed deterministic methods to study classical solvability of (1.1)–(1.4) for $N \leqslant 2$. Degond's techniques used a result of J. Lions [11] to study weak solvability of the linear problem (associated with each iterate approximate) in a L^2-setting. With enough smoothness imposed on the data, these weak solutions to the sequence of linear problems were shown to converge to a weak solution of the nonlinear problem. This solution could be regularized to lie in a suitable Sobolev space containing candidates for classical solutions. This regularization could be carried out, however, for all time only when $N = 1$ or 2.

The existence of classical solutions to the model discussed above with general data was addressed by H. D. Victory, Jr. and B. P. O'Dwyer in [13]. This work, however, delves into the question of global existence of weak solutions (1.1)–(1.4). At this point, it is appropriate to point out that weak solvability of kinetic models has been studied by R. DiPerna and P. L. Lions in [5], where global existence of weak solutions was announced for the Boltzmann equation under general assumptions on the collision operators and initial conditions. In the same work the authors indicated how the Fokker–Planck cases can be treated.

In our work, we consider a "mollified" or modified equation, which results from (1.2) by mollifying the Poisson kernel, $(-\lambda/\omega_N(N-2))\,|x|^{-(N-2)}$ in a δ-neighborhood of the origin. Weak solvability of the system (1.1)–(1.4) is defined in a manner similar to that of P. Lax for hyperbolic conservation laws [10]. We show, by using the techniques in [13], that for any $\delta > 0$, the mollified problem has a unique weak solution $f^\delta(t, x, v)$ under rather general conditions imposed on f_0. Conservation of mass or charge holds for these approximates. This property allows us to view $f^\delta(t, x, v)$ as the density or distribution of associated probability measures defined on the Borel sets of $\mathbb{R}^N \times \mathbb{R}^N$. The global (in time) existence of a weak solution of (1.1)–(1.4) is then shown by letting $\delta \to 0$ and using compactness arguments. These arguments are effected by using Prohorov's Theorem [1] characterizing weakly compact sets of probability measures. A crucial step in these arguments is the supposition that the kinetic energies

$$\int_{\mathbb{R}^{2N}} \int |v|^2 f^\delta(t, x, v)\, dv\, dx, \qquad \delta > 0,$$

remain bounded for all t, uniformly in δ. This is always true, as we shall see, in the plasma physical case with arbitrary data, but only for $N \leqslant 3$ in the stellar dynamical case. Global existence of weak solutions for the stellar dynamical models with $N > 3$ is, in general, not present for arbitrary data, as a counterexample will show.

In Section II, we present the needed notation and results for the mollified version of (1.1)–(1.4). In Section III, assuming boundedness of the kinetic energies for any arbitrarily bounded time interval, we show the existence of a sequence $\delta_n \to 0$ such that the measures, $\mu_t^{(n)}$, given by

$$\mu_t^{(n)}(\mathscr{B}) := \int_{\mathscr{B}} \int f^{\delta_n}(t, x, v)\, dv\, dx, \tag{1.5}$$

converge weakly to a limit measure μ_t for all t. We prove that this limiting measure possesses a density $f(t, \cdot, \cdot)$, which we show is a weak solution of (1.1)–(1.4) in Section IV. The last section contains some results for the stellar dynamical case. At this point, we remark that our analysis is an extension of that by R. Illner and H. Neunzert [9] for collisionless models to plasmas with collisional effects included via the Fokker–Planck term and the dynamical friction term.

II. Results for Mollified Problems

In this section, we define some notation and concepts from measure theory which will determine the context of our weak formulation of (1.1)–(1.4). We first start with basic notation. A point in phase space $\mathbb{R}^{2N}$ is denoted by $P = (x, v)$, $x \in \mathbb{R}^N$, $v \in \mathbb{R}^N$. The Euclidean norm on $\mathbb{R}^N$ and $\mathbb{R}^{2N}$ is denoted by $|x|$ or $|v|$ and by $|P|$, respectively.

We let $\mathscr{M}_B$ be the set of all finite Borel measures, defined on the Borel subsets of $\mathbb{R}^{2N}$, with total variation equal to B. By $\mathfrak{C}_b(\mathbb{R}^{2N})$ (or $\mathfrak{C}_b$), we denote the Banach space of all continuous, bounded functions on $\mathbb{R}^{2N}$ under the supremum norm; by $\mathfrak{C}_0^n(\mathbb{R}^{2N})$ (or $\mathfrak{C}_0^n$) we denote the space of all n-times continuously differentiable functions with compact support. By $L^\infty(\mathbb{R}^{2N})$ and $L^1(\mathbb{R}^{2N})$, we mean the Banach spaces of all essentially bounded and absolutely integrable functions respectively on $\mathbb{R}^{2N}$, with norms denoted by $\|f\|_\infty$ and $\|f\|_1$, for an element f.

We shall show the existence of a weak solution to (1.1)–(1.4) as the limit of mollified solutions f^δ in a weak sense commonly occurring in probability theory. The following definition describes the relevant topology on $\mathcal{M}_B$:

DEFINITION II.1. A sequence $\{\mu_n : n \in \mathbb{N}\}$ of measures $\mu_n \in \mathcal{M}_B$ is said to converge weakly to $\mu \in \mathcal{M}_B$, $\mu_n \xrightarrow{w} \mu$, if

$$\lim_{n \to \infty} \int_{\mathbb{R}^{2N}} \int \phi(x, v)\, d\mu_n(x, v) = \int_{\mathbb{R}^{2N}} \int \phi(x, v)\, d\mu(x, v) \tag{2.1}$$

for all $\phi \in \mathfrak{C}_b$.

Remark. As is well known $\mu_n \xrightarrow{w} \mu$ if, and only if, (2.1) holds for every $\phi \in \mathfrak{C}_0^0$. As every $\phi \in \mathfrak{C}_0^0$ can be approximated in the maximum norm by elements of $\mathfrak{C}_0^n$, it follows that $\mu_n \xrightarrow{w} \mu$ if (2.1) is true for every $\phi \in \mathfrak{C}_0^n$.

DEFINITION II.2. A set $\mathcal{S} \subset \mathcal{M}_B$ is termed tight (straff) if for any $\varepsilon > 0$, there is a compact set $\Omega \subset \mathbb{R}^{2N}$ such that, for all $\mu \in \mathcal{S}$,

$$\mu(\Omega) > B - \varepsilon.$$

Remark. Prohorov's Theorem [1] characterizes relatively compact subsets of $\mathcal{M}_B$ with respect to the topology described in Definition II.1: A *set $\mathcal{S} \subseteq \mathcal{M}_B$ is relatively compact if, and only if, it is tight.*

The following definition is needed to precisely describe the time dependence of weak solutions of (1.1)–(1.4):

DEFINITION II.3. The mapping $t \mapsto \mu_t \in \mathcal{M}_B$, $t \in [0, T]$, is weakly continuous if

$$t \to \int_{\mathbb{R}^{2N}} \int \phi(x, v)\, d\mu_t(x, v) \tag{2.2}$$

is continuous for any $\phi \in \mathfrak{C}_b$. Similarly, we say that a function f: $[0, T] \times \mathbb{R}^{2N} \to \mathbb{R}$ is weakly continuous with respect to $t \in [0, T]$ if the corresponding measures μ_t, with densities $f(t, \cdot, \cdot)$, are weakly continuous in t, i.e., the mapping

$$t \to \int_{\mathbb{R}^{2N}} \int f(t, x, v)\, \phi(x, v)\, dv\, dx \tag{2.3}$$

is continuous for any $\phi \in \mathfrak{C}_b$.

We are now in a position to state our concept of weak solvability of (1.1)–(1.4):

DEFINITION II.4. Let $T>0$ be arbitrary but fixed. The function $f\colon [0, T]\times \mathbb{R}^{2N}\to \mathbb{R}$ is termed a weak solution of (1.1)–(1.4) if:

(i) $f(t,\cdot)\in L^1_+(\mathbb{R}^{2N})$ (i.e., f is nonnegative for each t) and

$\|f(t,\cdot,\cdot)\|_1 = \|f_0\|_1$ (Conservation of mass or charge);

(ii) for $\mathbf{E}(t, x)$ defined by

$$\mathbf{E}(t,x) := -\frac{\lambda}{\omega_N}\int_{\mathbb{R}^N}\frac{x-y}{|x-y|^N}\int_{\mathbb{R}^N}[f(t,y,v)-N_0(y,v)]\,dv\,dy$$

(with λ defined in the paragraph following condition (1.4)), and for all $\phi\in \mathfrak{C}^2_0([0,T]\times\mathbb{R}^{2N})$, the equation

$$\int_0^T\left(\int_{\mathbb{R}^{2N}}\int f(t,x,v)\left[\frac{\partial\phi}{\partial t}(t,x,v)+v\cdot\nabla_x\phi(t,x,v)\right.\right.$$
$$\left.\left.+\ (\mathbf{E}(t,x)-\beta v)\cdot\nabla_v\phi(t,x,v)-\sigma\Delta_v\phi(t,x,v)\right]dv\,dx\right)dt$$
$$+\int_{\mathbb{R}^{2N}}\int\phi(0,x,v)\,f_0(x,v)\,dv\,dx=0 \tag{2.4}$$

holds, where $f_0\in L^1(\mathbb{R}^{2N})$ is the given initial electron distribution, and $N_0\in L^1(\mathbb{R}^{2N})$ is the distribution of the fixed ion background.

We now turn to an analysis of mollified problems. For this task we shall draw on results from [12, 13]. In order to define the mollified version of (1.1)–(1.4), we let $\tilde{\omega}(\xi)$ be any nonnegative function in the space $\mathfrak{S}$ of testing functions of rapid descent. Then let $\omega(x)$ be the inverse Fourier transform of $\tilde{\omega}$, which we assume has L^1-norm equal to unity, without loss of generality. Let $\delta>0$ be arbitrary and define the mollifier $\omega_\delta\colon \mathbb{R}^N\to\mathbb{R}$ by

$$\omega_\delta(x)=\frac{1}{\delta^N}\,\omega\left(\frac{x}{\delta}\right). \tag{2.5}$$

The mollified potential and field are generated by the kernels $u_\delta(x-y)$ and $k_\delta(x-y)$ defined respectively for any $x\in\mathbb{R}^N$ by

$$u_\delta(x) := \frac{-\lambda}{(N-2)\,\omega_N}\int_{\mathbb{R}^N}|x-y|^{-(N-2)}\,\omega_\delta(y)\,dy \tag{2.6}$$

and

$$k_\delta(x) := \frac{-\lambda}{\omega_N}\int_{\mathbb{R}^N}(x-y)\,|x-y|^{-N}\,\omega_\delta(y)\,dy. \tag{2.7}$$

Our assumptions on the problem data f_0 and N_0 are the following:

$$\text{(i)}\quad f_0, N_0 \in L^\infty(\mathbb{R}^{2N}), \tag{2.8}$$

$$\text{(ii)}\quad \int_{\mathbb{R}^{2N}}\int (|x|^2+|v|^2)\, f_0(x, v)\, dx\, dv < \infty, \tag{2.9}$$

$$\text{(iii)}\quad \int_{\mathbb{R}^{2N}}\int (|x|^2+|v|^2)\, N_0(x, v)\, dv\, dx < \infty. \tag{2.10}$$

The mollified version of (1.1)–(1.4) is the following:

$$\begin{aligned}\frac{\partial f^\delta}{\partial t}(t, x, v) + v\cdot\nabla_x f^\delta(t, x, v) + (\mathbf{E}^\delta(t, x) - \beta v)\cdot\nabla_v f^\delta(t, x, v)\\ = N\beta f^\delta(t, x, v) + \sigma\Delta_v f^\delta(t, x, v),\ (t, x, v)\in(0, \infty)\times\mathbb{R}^{2N},\end{aligned} \tag{2.11}$$

with

$$\begin{aligned}\mathbf{E}^\delta(t, v) &= \int_{\mathbb{R}^N} k_\delta(x-y)\int_{\mathbb{R}^N} [f^\delta(t, y, v) - N_0(y, v)]\, dv\, dy\\ &= -\nabla_x \mathbf{U}^\delta(t, x),\end{aligned} \tag{2.12}$$

where $\mathbf{U}^\delta(t, x)$ is the mollified potential given by

$$\mathbf{U}^\delta(t, x) = \int_{\mathbb{R}^N} u_\delta(x-y)\int_{\mathbb{R}^N} [f^\delta(t, y, v) - N_0(y, v)]\, dv\, dy. \tag{2.13}$$

The approximate distribution f^δ at $t=0$ is defined to be

$$f^\delta|_{t=0} = f_0. \tag{2.14}$$

The actual mathematical analysis of the mollified problem will parallel that of the unmollified problem treated in [13]. The appropriate equations for the various iterates to (2.11)–(2.14) will be the same as for the unmollified version. In the next few paragraphs, we shall briefly summarize the proof of global existence of solutions to mollified Vlasov–Poisson–Fokker–Planck models, referring occasionally to the analysis in [13].

In [13], the proof of existence of classical solutions to (1.1)–(1.4) hinges crucially on obtaining a priori estimates on the various iterates of the electric field, and on the spatial derivatives, on arbitrarily bounded time intervals. The failure to obtain such estimates for $N \geqslant 3$ means that only local existence of classical solutions for arbitrary data in these spatial (or momentum) dimensions can be deduced. On the other hand, such estimates on the iterates of the underlying field $\mathbf{E}_n^\delta(t, x)$ are immediate for the mollified setting, since arbitrary derivatives of $\omega_\delta(x)$ in (2.5), and hence of $u_\delta(x)$ and $k_\delta(x)$ defined by (2.6) and (2.7), are uniformly bounded on $\mathbb{R}^N$. Because of the mollification, we have $\mathbf{E}_n^\delta(t, \cdot) \in \mathfrak{C}_b^\infty(\mathbb{R}^N)$ for each $t \in (0, \infty) := R_+$. The next few paragraphs will indicate how these properties of $\mathbf{E}_n^\delta$ imply the global existence, smoothness, and uniqueness of solutions to (2.11)–(2.14).

The regularity of the $f_n^\delta(t, x, v)$ can be seen by the following argument. For $n = 0$, we have from (2.12) that $\mathbf{E}_0^\delta(t, x)$ is $\mathfrak{C}^\infty$ in all arguments. Accordingly, the analysis in [13, Sect. II] indicates that the resulting linear equation for the next iterate $f_1^\delta(t, x, v)$ possesses a nonnegative fundamental solution, denoted as $\Gamma_{\beta,\delta}^1(x, v, t, \xi, \nu, \tau)$; and the resulting $f_1^\delta(t, x, v)$ can be expressed via

$$f_1^\delta(t, x, v) = \int_{\mathbb{R}^{2N}} \int \Gamma_{\beta,\delta}^1(x, v, t, \xi, \nu, 0)\, f_0(\xi, \nu)\, d\xi\, d\nu. \tag{2.15}$$

This representation enables us to conclude that $f_1^\delta(t, \cdot, \cdot) \in \mathfrak{C}_b^\infty(\mathbb{R}^{2N})$ for each $t > 0$, and is, moreover, uniformly continuous in all variables for $t \geqslant \delta_0 > 0$, $(x, v) \in \mathbb{R}^{2N}$. The analysis carried out in [13, Sect. II] shows that the solution f_1^δ is unique in the class of functions having the regularity features described above, and taking on the initial data $f_0(x, v)$ in both the $L^1(\mathbb{R}^{2N})$- and $L^\infty(\mathbb{R}^{2N})$-senses as $t \to 0^+$.

Now defining $\mathbf{E}_1^\delta(t, x)$ by means of $f_1^\delta(t, x, v)$, substituted for $f^\delta(t, x, v)$ in (2.12), we see that for each $t \geqslant 0$, $\mathbf{E}_1^\delta(t, \cdot) \in \mathfrak{C}_b^\infty(\mathbb{R}^N)$ and indeed is continuous on $[0, \infty)$. Hence, a fundamental solution for the equation determining $f_2^\delta(t, x, v)$ exists and is nonnegative and $f_2^\delta(t, x, v)$ can be expressed by a formula similar to (2.15), which is a unique representation. The iterate $f_2^\delta(t, x, v)$ will possess the same regularity properties as $f_1^\delta(t, x, v)$ and takes the initial data in both the L^1- and L^∞-senses.

These deliberations can be extended to the other iterates by means of the fundamental solution constructed with the previous iterate. We can justify moving derivatives with respect to x and v of arbitrary orders inside the integral defining $f_n^\delta(t, x, v)$,

$$f_n^\delta(t, x, v) = \int_{\mathbb{R}^{2N}} \int \Gamma_{\beta,\delta}^n(x, v, t, \xi, \nu, 0)\, f_0(\xi, \nu)\, d\xi\, d\nu, \tag{2.16}$$

with the resulting integrals converging, since $\mathbf{E}_{n-1}^\delta(t, \cdot) \in \mathfrak{C}_b^\infty(\mathbb{R}^{2N})$, and f_n^δ is nonnegative and continuous in $t \in [0, \infty)$. This implies that we can differentiate arbitrarily often with respect to t for $t > 0$. All these observations manifest the fact that the Vlasov–Poisson–Fokker–Planck equation for each iterate is hypoelliptic, and the iterates thereof reside in $\mathfrak{C}_b^\infty((0, \infty) \times \mathbb{R}^{2N})$ (cf., e.g., [6]).

The next stage is to show that $\{f_n^\delta(t, \cdot, \cdot)\}$ is a Cauchy sequence in $L^1(\mathbb{R}^{2N}) \cap L^\infty(\mathbb{R}^{2N})$ for each t, and more precisely, that

$$\sup_{0 \leqslant \tau \leqslant t} \|f_n^\delta(\tau, \cdot, \cdot) - f_{n-1}^\delta(\tau, \cdot, \cdot)\|_1 \to 0 \qquad \text{as } n \to \infty, \tag{2.17}$$

along with

$$\sup_{0 \leqslant \tau \leqslant t} \|f_n^\delta(\tau, \cdot, \cdot) - f_{n-1}^\delta(\tau, \cdot, \cdot)\|_\infty \to 0 \qquad \text{as } n \to \infty. \tag{2.18}$$

To show (2.17), we merely mimic the proof of Proposition III.3 of [13] for the exact problem. Then we may immediately deduce that

$$\sup_{0 \leqslant \tau \leqslant t} \|\mathbf{E}_n^\delta(\tau, \cdot) - \mathbf{E}_{n-1}^\delta(\tau, \cdot)\|_\infty \to 0 \qquad \text{as } n \to \infty$$

which leads to (2.18) via [13, eq. (3.74)]. Letting $f^\delta(t, \cdot, \cdot)$ denote the limit of $f_n^\delta(t, \cdot, \cdot)$ in both the $L^\infty(\mathbb{R}^{2N})$- and $L^1(\mathbb{R}^{2N})$-senses, we can immediately see that $f^\delta(t, \cdot, \cdot) \geqslant 0$ and that the data f_0 are assumed as $t \to 0^+$ in both the L^1- and L^∞-senses.

We now discuss regularity in t. More precisely, we show that f^δ: $[0, T] \times \mathbb{R}^{2N}$ is weakly continuous with respect to t in any closed and bounded subinterval of $\mathbb{R}^+$. Toward this end, we observe that the following mapping

$$t \to \int_{\mathbb{R}^{2N}} \int \phi(x, v)\, f_n^\delta(t, x, v)\, dv\, dx \tag{2.19}$$

is continuous, $t \in [0, T]$, $\phi \in \mathfrak{C}_b(\mathbb{R}^{2N})$. This is easy to see: for $t \geqslant \delta_0 > 0$, this is obvious by the inherent continuity of each iterate for $t > 0$; for $t < \delta_0$, the continuity results from the fact that the data are assumed in both the L^1- and L^∞-senses. Because of the L^1-convergence of the iterates $f_n^\delta(t, \cdot, \cdot)$ to $f^\delta(t, \cdot, \cdot)$ for each t, we have that

$$\int_{\mathbb{R}^{2N}} \int \phi(x, v)\, f_n^\delta(t, x, v)\, dv\, dx \to \int_{\mathbb{R}^{2N}} \int \phi(x, v)\, f^\delta(t, x, v)\, dv\, dx \tag{2.20}$$

pointwise in $t \in [0, T]$. In order to show that the latter quantity is continuous in t, we need only show that the sequence

$$\psi_n^\delta(t) = \int_{\mathbb{R}^{2N}} \int f_n^\delta(t, x, v)\, \phi(x, v)\, dv\, dx$$

is equicontinuous in t. We shall omit the proof of this result; it is a trivial adaptation of the proof of Lemma III.3, which is provided in full detail as it underpins the convergence analysis as $\delta \to 0^+$. In the mollified setting, though, the estimates do not have to be as delicate as those in Section III, since here we are concerned with a fixed $\delta > 0$. At this juncture, we deem it appropriate to remark that the limiting function f^δ is a weak solution of (2.11)–(2.14) in the sense of Lax. The proof of this fact also is a trivial adaptation of the proof given in Theorem 2 for the limiting function of the $f^\delta(t, \cdot, \cdot)$ as $\delta \to 0$.

We are now in a position to provide the analogue of the "continuity equation" of Vlasov–Poisson transport. In order to derive this, we first note that each iterate f_n^δ can be represented as in (2.16) in terms of the fundamental solution of (2.11) constructed with $\mathbf{E}_{n-1}^\delta(t, x)$. The representation of f^δ in terms of a "continuity equation" then results from the convergence of the iterate approximates to the electric field and of their derivatives, i.e.,

$$\sup_{0 \leqslant \tau \leqslant t} \|\mathbf{E}_n^\delta(\tau, \cdot) - \mathbf{E}^\delta(\tau, \cdot)\|_\infty \to 0 \qquad \text{as } n \to \infty \tag{2.21}$$

and of

$$\sup_{0 \leqslant \tau \leqslant t} \|\nabla_x \mathbf{E}_n^\delta(\tau, \cdot) - \nabla_x \mathbf{E}_{n+1}^\delta(\tau, \cdot)\|_{\infty,1} \to 0 \qquad \text{as } n \to \infty \tag{2.22}$$

(where the notation $\nabla_x \mathbf{E}_n^\delta(t, x)$, and the matrix norm $\|—\|_{\infty,1}$, have been defined in [13, Sect. II]). The procedure in [13, Section II], or in [12], shows that the fundamental solution to (2.11) constructed with the limiting field $\mathbf{E}^\delta(t, x)$, and denoted by $\Gamma_{\beta,\delta}(x, v, t, \xi, v, \tau)$, has the property that

$$\begin{aligned} \sup_{0 \leqslant \tau \leqslant t} \Bigg\| \int_{\mathbb{R}^{2N}} \int & (\Gamma_{\beta,\delta}(x, v, \tau, \xi, v, 0) \\ & - \Gamma_{\beta,\delta}^n(x, v, \tau, \xi, v, 0)) f_0(\xi, v) \, d\xi \, dv \Bigg\|_\infty \to 0. \end{aligned} \tag{2.23}$$

This follows from an easy Gronwall argument, applied to the following representation of (2.23),

$$\begin{aligned} &\int_{\mathbb{R}^{2N}} \int (\Gamma_{\beta,\delta}(x, v, t, \xi, v, 0) - \Gamma_{\beta,\delta}^n(x, v, t, \xi, v, 0)) f_0(\xi, v) \, d\xi \, dv \\ &\quad = \int_0^t \int_{\mathbb{R}^{2N}} \int \nabla_{v'} G_\beta(x, v, t, \xi', v', \tau')[\mathbf{E}^\delta(\tau', \xi') - \mathbf{E}_n^\delta(\tau', \xi')] \\ &\qquad \cdot \int_{\mathbb{R}^{2N}} \int \Gamma_{\beta,\delta}(\xi', v', \tau', \xi, v, 0) f_0(\xi, v) \, d\xi \, dv \, d\xi' \, dv' \, d\tau' \\ &\qquad + \int_0^t \int_{\mathbb{R}^{2N}} \int \nabla_{v'} G_\beta(x, v, t, \xi', v', \tau') \, \mathbf{E}_n^\delta(\tau', \xi') \\ &\qquad \cdot \int_{\mathbb{R}^{2N}} \int (\Gamma_{\beta,\delta}(\xi', v', \tau', \xi, v, 0) \\ &\qquad - \Gamma_{\beta,\delta}^n(\xi', v', \tau', \xi, v, 0)) f_0(\xi, v) \, d\xi \, dv \, d\xi' \, dv' \, d\tau'. \end{aligned} \tag{2.24}$$

Here the function $G_\beta(x, v, t, \xi', v', \tau')$ denotes the fundamental solution to the field-free version of (2.11), and is given by

$$G_\beta(x, v, t, \xi, v, \tau)$$
$$= \left[\frac{\beta \exp(\beta(t-\tau))}{4\pi\sigma \sqrt{\dfrac{\exp(2\beta(t-\tau))-1}{2\beta}(t-\tau) - \dfrac{(\exp(\beta(t-\tau))-1)^2}{\beta^2}}}\right]^N$$
$$\exp\left\{-\frac{(\beta x - \beta\xi + v - v)^2}{4\sigma(t-\tau)}\right\}$$
$$\cdot \exp\left\{-\frac{\left|\dfrac{\exp(\beta(t-\tau))-1}{(t-\tau)}\left(x-\xi+\dfrac{v-v}{\beta}\right)+(v - v\exp(\beta(t-\tau))\right|^2}{4\sigma\left[\dfrac{\exp(2\beta(t-\tau))-1}{2\beta} - \dfrac{(\exp(\beta(t-\tau))-1)^2}{\beta^2(t-\tau)}\right]}\right\}. \tag{2.25}$$

By uniqueness of the limits, then, we have that

$$f^\delta(t, x, v) = \int_{\mathbb{R}^{2N}}\int \Gamma_{\beta,\delta}(x, v, t, \xi, v, 0)\, f_0(\xi, v)\, d\xi\, dv, \qquad t > 0. \tag{2.26}$$

Because of the existence of higher-order spatial and velocity derivatives of $\mathbf{E}^\delta(t, x)$, $t \geqslant 0$, easily shown by standard arguments, and of the existence of arbitrary time derivatives, $t > 0$, the fundamental solution constructed with the limiting electric field will be arbitrarily differentiable with respect to $t > 0$ and $(x, v) \in \mathbb{R}^{2N}$. This means that $f^\delta(t, x, v)$, via its representation in terms of a continuity equation, is likewise smooth for $(x, v) \in \mathbb{R}^{2N}$, $t > 0$. We refer the reader to [12] or [13] for a more thorough discussion of these facts. Therefore, $f^\delta(t, \cdot, \cdot)$ is de-facto a classical solution of (2.11) assuming its data in the L^∞- and L^1-senses. The uniqueness of mollified solutions follows from a rather straightforward adaptation of the uniqueness proof for classical solutions to (1.1)–(1.4) (cf., e.g., [13, Theorem III.3]).

The next few comments indicate how the mollified Vlasov–Poisson–Fokker–Planck dynamics preserves the finiteness of the initial kinetic energy and moment of inertia. In other words, we wish to conclude that

$$\int_{\mathbb{R}^{2N}}\int (|x|^2 + |v|^2)\, f^\delta(t, x, v)\, dx\, dv < \infty$$

whenever f_0 possesses this property. Toward this end, we can provide the following estimate of $f^{\delta}(t, x, v)$ is terms of the solution to a field free problem with Cauchy data f_0,

$$\begin{aligned} f^{\delta}(t, x, v) &:= \int_{\mathbb{R}^{2N}}\int \Gamma_{\beta,\delta}(x, v, t, \xi, \nu, 0)\, f_0(\xi, \nu)\, d\xi\, d\nu \\ &\leqslant \int_{\mathbb{R}^{2N}}\int G_{\beta}(x, v, t, \xi, \nu, 0)\, f_0(\xi, \nu)\, d\xi\, d\nu \\ &\quad + \sum_{l=0}^{\infty} \left[\frac{M(\alpha, \beta, \sigma)}{\alpha^{2N}}\right]^{l+1} \frac{\pi^{l/2}}{\Gamma(l/2+1)} \\ &\quad \cdot \int_0^t \Big[\sup_{0 \leqslant \tau'' \leqslant \tau'} \|\mathbf{E}^{\delta}(\tau'', \cdot)\|_{\infty}\Big]^{l+1} (t-\tau')^{-1/2}\, (\tau')^{l/2}\, d\tau' \\ &\quad \cdot \int_{\mathbb{R}^{2N}}\int G_{\beta}(\alpha x, \alpha v, t, \alpha\xi, \alpha\nu, 0)\, f_0(\xi, \nu)\, d\xi\, d\nu, \end{aligned} \tag{2.27}$$

where Γ here denotes the well-known Gamma function; $M(\alpha, \beta, \sigma)$ is a constant depending on the arguments indicated but independent of δ, and α is any number between 0 and 1. From this estimate it is clear that we must examine the quantity

$$\int_{\mathbb{R}^{2N}}\int f_0(\xi, \nu) \int_{\mathbb{R}^{2N}}\int (|x|^2 + |v|^2)\, G_{\beta}(x, v, t, \xi, \nu, 0)\, dx\, dv\, d\xi\, d\nu. \tag{2.28}$$

A rather tedious computation yields that

$$\begin{aligned} &\int_{\mathbb{R}^{2N}}\int (|x|^2 + |v|^2)\, G_{\beta}(x, v, t, \xi, \nu, 0)\, dx\, dv \\ &\quad = \frac{4\sigma t N}{2\beta^2}\left\{\left(1 - \frac{(1-e^{-\beta t})}{\beta t}\right)^2 + \frac{(1-e^{-\beta t})^2}{t^2} + (1+\beta^2)\frac{e^{-2\beta t}\zeta(t)}{t^2}\right\} \\ &\qquad + \left|\xi + \frac{\nu}{\beta}(1-e^{-\beta t})\right|^2 + |\nu e^{-\beta t}|^2, \end{aligned} \tag{2.29}$$

where

$$\zeta(t) := \frac{\exp(2\beta t) - 1}{2\beta}\, t - \frac{(\exp \beta t - 1)^2}{\beta^2}.$$

The desired result is immediate.

We recall that the potential energy of the mollified system can be expressed in terms of a mollified potential by

$$\mathfrak{PE}^{\delta}(t) := \int_{\mathbb{R}^{2N}}\int u_{\delta}(x-y)\, \rho^{\delta}(t, y)\, \rho^{\delta}(t, x)\, dx\, dy, \tag{2.30}$$

where

$$\rho^{\delta}(t, y) = \int_{\mathbb{R}^N} [f^{\delta}(t, y, v) - N(y, v)]\, dv \tag{2.31}$$

denotes the charge or mass density. We also define the kinetic energy of the mollified system by

$$\mathfrak{KE}^{\delta}(t) := \int_{\mathbb{R}^{2N}}\int |v|^2 f^{\delta}(t, x, v)\, dv\, dx, \tag{2.32}$$

and the moment of inertia by

$$h^{\delta}(t) := \int_{\mathbb{R}^{2N}}\int |x|^2 f^{\delta}(t, x, v)\, dv\, dx. \tag{2.33}$$

The smoothness and regularity of solutions to the mollified system (2.11)–(2.14) allow us to rigorously justify the formal integration of (2.11) to obtain the following equations relating these quantities:

$$\frac{d}{dt}\mathfrak{PE}^{\delta}(t) + \frac{d}{dt}\mathfrak{KE}^{\delta}(t) = -2\beta\mathfrak{KE}^{\delta}(t) + 2N\sigma \|f_0\|_1 \tag{2.34}$$

$$\frac{dh_{\delta}(t)}{dt} = 2\int_{\mathbb{R}^{2N}}\int (x \cdot v)\, f^{\delta}(t, x, v)\, dv\, dx \tag{2.35}$$

$$\frac{d^2 h_{\delta}(t)}{dt^2} + \beta\frac{dh_{\delta}(t)}{dt} = 2\mathfrak{KE}^{\delta}(t) + (N-2)\,\mathfrak{PE}^{\delta}(t). \tag{2.36}$$

Because of the sign of λ in (2.6) for the plasma physical setting, and because of the nonnegativity of the Fourier transform of $\omega_{\delta}(x)$, a simple Fourier transform argument enables us to deduce $u_{\delta}(x-y)$ is a positive definite kernel, and the bilinear form defining $\mathfrak{PE}^{\delta}(t)$ is nonnegative. Accordingly, we can immediately deduce from (2.34) that, for any dimension N,

$$\mathfrak{KE}^{\delta}(t) \leqslant 2N\sigma \|f_0\|_1\, t + \mathfrak{KE}^{\delta}(0) + \mathfrak{PE}^{\delta}(0) \tag{2.37}$$

which can be bounded on arbitrary time intervals independently of δ owing to the integrability and boundedness of f_0. In the stellar dynamical setting, the potential energy is always negative, and we must employ subtle arguments, due to E. Horst [8], in order to prove (2.37) for $N=3$. In the next few sections, we shall exploit (2.34)–(2.36) in order to produce a weakly convergent sequence $f^{\delta_n}(t, x, v)$ as $\delta_n \to 0^+$.

III. Convergence of Solutions to Mollified Problems

In this section, we exploit the uniform boundedness of the kinetic energies to deduce the existence of a sequence $\delta_n \to 0^+$ such that the associated measures, $\mu_t^{(n)}$, given by

$$\mu_t^{(n)}(\mathscr{B}) := \int_{\mathscr{B}} \int f^{\delta_n}(t, x, v)\, dv\, dx,$$

converges weakly to a limit measure μ_t for all t. We shall subsequently use some properties of the kernel $-\lambda\omega_N^{-1}(x-y)\,|x-y|^{-N}$ to show that the limiting measure μ_t possesses a density which we show is a weak solution to (1.1)–(1.4).

The primary goal of this section is the proof of the following:

Theorem 1. *Let f_0 and N_0 satisfy* (2.8)–(2.10). *Suppose that the kinetic energies*

$$\mathfrak{K}\mathfrak{E}^{\delta}(t) := \int_{\mathbb{R}^{2N}} \int |v|^2 f^{\delta}(t, x, v)\, dv\, dx \tag{3.1}$$

associated with the solutions f^{δ} of the mollified problems (2.11)–(2.14) *are uniformly bounded on arbitrarily bounded time intervals, $[0, T]$, $T < \infty$, with bounds independent of δ, but possibly depending on T. Then there is a sequence $\{\delta_n : n \in \mathbb{N}\}$, $\delta_n \to 0^+$ such that the measures $\mu_t^{(n)}$, defined by* (1.5), *converge weakly to a measure $\mu_t \in \mathscr{M}_B$, $B := \|f_0\|_1$. The mapping from $[0, T]$, $T < \infty$ to $\mathscr{M}_B$, given by*

$$t \in [0, T] \to \mu_t, \tag{3.2}$$

is weakly continuous and μ_t is absolutely continuous with respect to the Lebesgue measure.

We shall organize the proof in the form of lemmas, starting with the generation of a convergent sequence $\{\mu_t^{(n)} : \delta_n \to 0^+\}$:

Lemma III.1. *The set of measures $\{\mu_t^{\delta} : t \in [0, T], \delta > 0\}$ is tight in $\mathscr{M}_B$, $B := \|f_0\|_1$.*

Proof. The definition of B as $\|f_0\|_1$ is easily deduced, and is a restatement of the conservation of mass or charge,

$$\int_{\mathbb{R}^{2N}} \int f^{\delta}(t, x, v)\, dx\, dv = \int_{\mathbb{R}^{2N}} \int f_0(x, v)\, dx\, dv,$$

a consequence of the underlying mollified problem (2.11)–(2.14). The proof of the lemma uses properties of the moment of inertia. From (2.35), and an adroit use of the Schwarz Inequality, we can deduce that

$$\left|\frac{dh_\delta}{dt}\right| \leqslant 2\left[\int_{\mathbb{R}^{2N}}\int |v|^2 f^\delta(t,x,v)\,dv\,dx\right]^{1/2}\left[\int_{\mathbb{R}^{2N}}\int |x|^2 f^\delta(t,x,v)\,dv\,dx\right]^{1/2}, \tag{3.3}$$

or that

$$|h'_\delta(t)| \leqslant C(T)\,h_\delta^{1/2}(t),\ C(t) = 2[2N\sigma\,\|f_0\|_1\,t + \mathfrak{KE}(0) + \mathfrak{PE}^\delta(0)]^{1/2}. \tag{3.4}$$

We immediately deduce that

$$h_\delta(t) \leqslant 2[2N\sigma\,\|f_0\|_1\,t^3 + (\mathfrak{KE}(0) + \mathfrak{PE}^\delta(0))\,t^2 + h_d(0)] := C_t, \tag{3.5}$$

a constant independent of δ.

This has the profound implication that the set

$$\{\mu_t^\delta : \delta > 0,\ t \in [0, T]\}$$

is tight (straff) in $\mathcal{M}_B$, and hence is relatively compact in $\mathcal{M}_B$, $B = \|f_0\|_1$. Let $\varepsilon > 0$ be arbitrary and select $R > 0$ so large that $[C_T + 2N\sigma\,\|f_0\|_1\,T + \mathfrak{PE}^\delta(0) + \mathfrak{KE}(0)]/R^2 < \varepsilon$. Now let $K_{N,R} = \{(x,v) : |x|^2 + |v|^2 \leqslant R^2\}$, and observe that

$$\begin{aligned}
R^2\mu_t^\delta(\mathbb{R}^{2N}\backslash K_{N,R}) &= R^2\int_{\mathbb{R}^{2N}\backslash K_{N,R}}\int f^\delta(t,x,v)\,dx\,dv \\
&\leqslant \int_{\mathbb{R}^{2N}\backslash K_{N,R}}\int (|x|^2 + |v|^2)\,f^\delta(t,x,v)\,dv\,dx \\
&\leqslant \int_{\mathbb{R}^{2N}}\int (|x|^2 + |v|^2)\,f^\delta(t,x,v)\,dv\,dx \\
&\leqslant h_\delta(t) + 2N\sigma\,\|f_0\|_1\,t + \mathfrak{PE}^\delta(0) + \mathfrak{KE}(0) \\
&\leqslant C_T + 2N\sigma\,\|f_0\|_1\,T + \mathfrak{PE}^\delta(0) + \mathfrak{KE}(0) \leqslant \varepsilon R^2.
\end{aligned} \tag{3.6}$$

So

$$\mu_t^\delta(\mathbb{R}^{2N}\backslash K_{N,R}) < \varepsilon$$

for sufficiently large R and all $\delta > 0$. The proof is complete.

Let, now, δ_n be any selected sequence of δ's such that $\delta_n \downarrow 0$ for each t. Lemma III.1 allows us to utilize Prohorov's Theorem to deduce the existence of a subsequence $n_k(t)$ such that

$$\mu_t^{\delta_{n_k(t)}} := \mu_t^{(n_k(t))} \xrightarrow{w} \mu_t \in \mathcal{M}_B. \tag{3.7}$$

If $\mathscr{T}' := \{t_\nu : \nu \in \mathbb{N}\}$ denotes a countable dense subset of $[0, T]$, then the *Cantor Diagonalization Procedure* assures us the existence of a monotone sequence again labeled as $\{\delta_n : n \in \mathbb{N}\}$, with $\delta_n \downarrow 0$ such that

$$\mu_{t_\nu}^{(n)} \xrightarrow{w} \mu_{t_\nu} \in \mathscr{M}_B, \qquad t_\nu \in \mathscr{T}'. \tag{3.8}$$

The task before us now is to show that this convergence is maintained for arbitrary t, and that the limiting measure is weakly continuous in t. A critical step in this direction is the following lemma which indicates that the contributions to the mollified forces, from particles whose positions and momenta reside in any selected compactum of $\mathbb{R}^{2N}$, remain bounded uniformly in δ_n. More precisely, we have:

LEMMA III.2. *Let $\mathscr{G} \subset \mathbb{R}^{2N}$ be compact, and define $k_n := k_{\delta_n}$, $n \in \mathbb{N}$. Then the quantities*

$$\chi_n(t, y) := \int_{\mathscr{G}} \int k_n(x-y) f^{\delta_n}(t, x, v)\, dv\, dx \tag{3.9}$$

are uniformly bounded with respect to $y \in \mathbb{R}^N$, $t \in [0, T]$, $n \in \mathbb{N}$, with bounds depending at most on the dimension N, but independent of δ_n.

Proof. Let $R > 0$ be such that

$$\mathscr{G} \subset \{(x, v) : |x| \leqslant R, |v| \leqslant R\}.$$

By the Fubini–Tonelli Theorem,

$$\begin{aligned} |\chi_n(t, y)| &\leqslant \int_{\mathscr{G}} \int |k_n(x-y)|\, f^{\delta_n}(t, x, v)\, dv\, dx \\ &\leqslant \int_{|x| \leqslant R} \left(\int_{|v| \leqslant R} |k_n(x-y)|\, f^{\delta_n}(t, x, v)\, dv \right) dx \\ &= \int_{|x| \leqslant R} |k_n(x-y)|\, \tilde{\rho}^{\delta_n}(t, x)\, dx, \end{aligned} \tag{3.10}$$

where

$$\tilde{\rho}^{\delta_n}(t, x) := \int_{\{v : |v| \leqslant R\}} f^{\delta_n}(t, x, v)\, dv.$$

Now

$$k_n(x-y) := \frac{-\lambda}{\omega_N} \int_{\mathbb{R}^N} (x-y-\xi)^{-N}\, \omega_{\delta_n}(\xi)\, d\xi,$$

and hence

$$|k_n(x-y)| \leqslant \frac{\sqrt{N}\,|\lambda|}{\omega_N} \int_{\mathbb{R}^N} |x-z|^{-(N-1)}\, |\omega_{\delta_n}(z-y)|\, dz.$$

We have, as a result,

$$\begin{aligned}
&|\chi_n(t, y)| \\
&\leqslant \sqrt{N}\,|\lambda|\,\omega_N^{-1} \int_{\{x:|x| \leqslant R\}} \left(\int_{\mathbb{R}^N} |x-z|^{-(N-1)}\, |\omega_{\delta_n}(z-y)|\, dz \right) \tilde{\rho}^{\delta_n}(t, x)\, dx \\
&\leqslant \sqrt{N}\,|\lambda|\,\omega_N^{-1} \int_{\mathbb{R}^N} |\omega_{\delta_n}(z-y)| \left(\int_{\mathbb{R}^N} |x-z|^{-(N-1)}\, \tilde{\rho}^{\delta_n}(t, x)\, dx \right) dz \\
&\leqslant \sqrt{N}\,|\lambda|\,\omega_N^{-1} \int_{\mathbb{R}^N} |\omega_{\delta_n}(z-y)| \left(\int_{\mathbb{R}^N} |z-x|^{-(N-1)} \tilde{\rho}^{\delta_n}(t, x)\, dx \right) dz. \qquad (3.11)
\end{aligned}$$

From [8], we can estimate

$$\begin{aligned}
&\sup_{0 \leqslant \tau \leqslant t \leqslant T} \left\| \int_{\mathbb{R}^N} |z-x|^{-(N-1)}\, \tilde{\rho}^{\delta_n}(\tau, x)\, dx \right\|_{\infty} \\
&\quad \leqslant C\left(N, N-1, \infty, \frac{N+2}{N} \right) \sup_{0 \leqslant \tau \leqslant t \leqslant T} \|\tilde{\rho}^{\delta_n}(\tau, \cdot)\|_{\infty}^{1-(N+2)/N^2} \\
&\quad\quad \cdot \sup_{0 \leqslant \tau \leqslant t \leqslant T} \|\tilde{\rho}^{\delta_n}(\tau, \cdot)\|_{(N+2)/N}^{(N+2)/N^2}, \qquad (3.12)
\end{aligned}$$

with

$$\begin{aligned}
&\sup_{0 \leqslant \tau \leqslant t \leqslant T} \|\tilde{\rho}^{\delta_n}(\tau, \cdot)\|_{(N+2)/N} \\
&\quad \leqslant \left(1 + \left(\frac{\omega_N}{N} \right) \right) \sup_{0 \leqslant \tau \leqslant t \leqslant T} \|f^{\delta_n}(\tau, \cdot, \cdot)\|_{\infty}^{2/(N+2)} \\
&\quad\quad \cdot \sup_{0 \leqslant \tau \leqslant t} \left(\int_{\mathbb{R}^{2N}}\!\!\int |v|^2 f^{\delta_n}(\tau, x, v)\, dv\, dx \right)^{N/(N+2)}, \qquad N \geqslant 3. \qquad (3.13)
\end{aligned}$$

Hence, we have the following

$$\begin{aligned}
&\sup_{0 \leqslant \tau \leqslant t \leqslant T} \left\| \int_{\mathbb{R}^N} |z-x|^{-(N-1)}\, \tilde{\rho}^{\delta_n}(\tau, x)\, dx \right\|_{\infty} \\
&\quad \leqslant C\left(N, N-1, \infty, \frac{N+2}{N}, f_0, T \right) \\
&\quad\quad \sup_{0 \leqslant \tau \leqslant t \leqslant T} \|\tilde{\rho}^{\delta_n}(\tau, \cdot)\|_{\infty}^{1-(N+2)/N^2}. \qquad (3.14)
\end{aligned}$$

We obtain the conclusion of the lemma by using the rather straightforward bound in (3.14),

$$\sup_{0 \leqslant \tau \leqslant t \leqslant T} \|\tilde{\rho}^{\delta_n}(t, \cdot)\|_\infty \leqslant \|f_0\|_\infty \, e^{N\beta T} \omega_N N^{-1} R^N. \tag{3.15}$$

Here, we have used the fact that $\omega_{\delta_n} \in L^1(\mathbb{R}^N)$, since $\omega \in \mathfrak{S}$. This completes the proof.

LEMMA III.3. *Let $\phi \in \mathfrak{C}_0^2$. Then the set of functions,*

$$t \to \psi_n(t) := \int_{\mathbb{R}^{2N}} \int \phi(x, v) \, f^{\delta_n}(t, x, v) \, dv \, dx \tag{3.16}$$

is equicontinuous on $[0, T]$.

Proof. Consider the quantity

$$\begin{aligned}
\psi_n(t) - \psi_n(s) &= \int_{\mathbb{R}^{2N}} \int \phi(x, v)[f^{\delta_n}(t, x, v) - f^{\delta_n}(s, x, v)] \, dv \, dx \\
&= \int_{\mathbb{R}^{2N}} \int \phi(x, v) \int_s^t \left(\frac{\partial}{\partial t} f^{\delta_n}\right)(\tau, x, v) \, d\tau \, dv \, dx \\
&= \int_s^t \int_{\mathbb{R}^{2N}} \int (-v \cdot \nabla_x f^{\delta_n}(\tau, x, v) - (\mathbf{E}^{\delta_n}(\tau, x) - \beta v) \\
&\qquad \cdot \nabla_v f^{\delta_n}(\tau, x, v) + N\beta f^{\delta_n}(\tau, x, v) \\
&\qquad + \sigma \Delta_v f^{\delta_n}(\tau, x, v)) \, \phi(x, v) \, dv \, dx \, d\tau \\
&= \int_s^t \int_{\mathbb{R}^{2N}} \int [v \cdot \nabla_x \phi(x, v) + (\mathbf{E}^{\delta_n}(\tau, x) - \beta v) \\
&\qquad \cdot \nabla_v \phi(x, v) + \sigma \Delta_v \phi(x, v)] \, f^{\delta_n}(\tau, x, v) \, dv \, dx \, d\tau.
\end{aligned}$$

Let now $\mathscr{G}$ denote the support of ϕ such that

$$\mathscr{G} \subset \{(x, v) \colon |x| \leqslant R, \, |v| \leqslant R\}.$$

Then we write

$$\begin{aligned}
\psi_n(t) - \psi_n(s) &= \int_s^t \int_{\mathscr{G}} \int [v \cdot \nabla_x \phi(x, v) + (\mathbf{E}^{\delta_n}(\tau, x) - \beta v) \\
&\qquad \cdot \nabla_v \phi(x, v) + \sigma \Delta_v \phi(x, v)] \, f^{\delta_n}(\tau, x, v)] \, dv \, dx \, d\tau \\
&\leqslant \sup_{(x, v) \in \mathscr{G}} \{[|\nabla_x \phi(x, v)|^2 + |\nabla_v \phi(x, v)|^2]^{1/2} + |\Delta_v \phi(x, v)|\} \\
&\qquad \cdot \left(\int_s^t \int_{\mathscr{G}} \int ((1 + \beta) \, |v| + \sigma) \, f^{\delta_n}(\tau, x, v) \, dv \, dx \, d\tau \right. \\
&\qquad \left. + \int_s^t \int_{\mathscr{G}} \int |\mathbf{E}^{\delta_n}(\tau, x)| \, f^{\delta_n}(\tau, x, v) \, dv \, dx \, d\tau \right).
\end{aligned} \tag{3.18}$$

We estimate various terms in (3.18): we observe that

$$\int_s^t \int_{\mathscr{G}} \int ((1+\beta)\,|v| + \sigma)\, f^{\delta_n}(\tau, x, v)\, dv\, dx \leqslant ((1+\beta)\, R + \sigma)\, B\, |t-s|, \tag{3.19}$$

and

$$\begin{aligned}
&\int_s^t \int_{\mathscr{G}} \int |\mathbf{E}^{\delta_n}(\tau, x)|\, f^{\delta_n}(\tau, x, v)\, dv\, dx\, d\tau \\
&\leqslant \int_s^t \int_{\mathscr{G}} \int \left| \int_{\mathbb{R}^N} k_n(x-y)\, \rho^{\delta_n}(\tau, y)\, dy \right| f^{\delta_n}(\tau, x, v)\, dv\, dx\, d\tau \\
&\leqslant \int_s^t \int_{\mathbb{R}^{2N}} \int \left(\int_{\mathscr{G}} \int |k_n(x-y)|\, f^{\delta_n}(\tau, x, v)\, dv\, dx \right) f^{\delta_n}(\tau, y, v')\, dv'\, dy\, d\tau \\
&\quad + \int_s^t \int_{\mathbb{R}^{2N}} \int \left(\int_{\mathscr{G}} \int |k_n(x-y)|\, f^{\delta_n}(\tau, x, v)\, dv\, dx \right) N_0(y, v')\, dv'\, dy\, d\tau \\
&\leqslant \left(\sup_{(t, y) \in [0, T] \times \mathbb{R}^N} \int_{\mathscr{G}} \int |k_n(x-y)|\, f^{\delta_n}(t, x, v)\, dv\, dx \right) \\
&\quad \cdot (\|f_0\|_1 + \|N_0\|_1)\, |t-s|.
\end{aligned}$$

Summarizing, we have

$$\begin{aligned}
&|\psi_n(t) - \psi_n(s)| \\
&\quad \leqslant \sup_{(x,v) \in \mathscr{G}} \{ (|\nabla_x \phi(x, v)|^2 + |\nabla_v \phi(x, v)|^2)^{1/2} + |\Delta_v \phi(x, v)| \} \\
&\qquad \cdot \Big\{ ((1+\beta)\, R + \sigma)\, B \\
&\qquad + \left(\sup_{(t, y) \in [0, T] \times \mathbb{R}^N} \int_{\mathscr{G}} \int |k_n(x-y)\,|\, f^{\delta_n}(t, x, v)\, dv\, dx \right) \\
&\qquad \cdot (\|f_0\|_1 + \|N_0\|_1) \Big\}\, |t-s|.
\end{aligned} \tag{3.20}$$

The desired equicontinuity follows.

It is well known that if the sequence of functions $\{\psi_n : n \in \mathbb{N}\}$ is equicontinuous on $[0, T]$ and pointwise convergent on a dense subset thereof, then it converges uniformly on $[0, T]$. Since $\psi_n(t) \to \int_{\mathbb{R}^{2N}} \int \phi(x, v)\, d\mu_t(x, v)$ for $t \in \mathscr{T}'$, then the preceding lemma assures us that $\psi_n(t) = \int_{\mathbb{R}^{2N}} \int \phi(x, v)\, d\mu_t^{(n)}(x, v)$ converges uniformly on $[0, T]$, if $\phi \in \mathfrak{C}_0^2$. By

Prohorov's Theorem, there is to each $t \notin \mathcal{T}'$, a subsequence n_k and a measure $\mu_t \in \mathcal{M}_B$ such that

$$\mu_t^{(n_k)} \xrightarrow{w} \mu_t \in \mathcal{M}_B,\ t \notin \mathcal{T}'.$$

From the convergence of $\psi_n(t)$, we conclude

$$\lim_{n \to \infty} \psi_n(t) = \lim_{k \to \infty} \psi_{n_k}(t) = \int_{\mathbb{R}^{2N}} \int \phi(x, v)\, d\mu_t(x, v), \qquad \phi \in \mathfrak{C}_0^2.$$

To every $t \in [0, T]$, there is a $\mu_t \in \mathcal{M}_B$ with

$$\int_{\mathbb{R}^{2N}} \int \phi(x, v)\, d\mu_t^{(n)}(x, v) \to \int_{\mathbb{R}^{2N}} \int \phi(x, v)\, d\mu_t(x, v),$$

i.e., $\mu_t^{(n)} \xrightarrow{w} \mu_t$.

The function $t \to \mu_t \in \mathcal{M}_B$ is weakly continuous. Let t_k be an arbitrary sequence in $[0, T]$, with limit t. As $\psi := \lim_{n \to \infty} \psi_n$ is continuous (as uniform limit of continuous functions), it follows that

$$\psi(t_k) := \int_{\mathbb{R}^{2N}} \int \phi(x, v)\, d\mu_{t_k} \to \psi(t)$$

as $k \to \infty$. Hence, $\mu_{t_k} \xrightarrow{w} \mu_t$.

We summarize the preceding remarks:

COROLLARY. *To every $t \in [0, T]$, there is a $\mu_t \in \mathcal{M}_B$ so that*

$$\mu_t^{(n)} \xrightarrow{w} \mu_t. \tag{3.21}$$

The function

$$t \to \mu_t, \qquad t \in [0, T], \tag{3.22}$$

is weakly continuous.

It remains to prove the absolute continuity of $\mu_t \in \mathcal{M}_B$. It is not obvious that our limiting measure $\mu_t \in \mathcal{M}_B$ possesses a density residing in $L^1(\mathbb{R}^{2N}) \cap L^\infty(\mathbb{R}^{2N})$, even though it is the weak limit of measures $\mu_t^{(n)}$ absolutely continuous with respect to the Lebesgue measure. The existence of the density is fundamentally due to the behavior of mollified forces. More precisely, we have:

LEMMA III.4. *The limiting measure μ_t, $t \in [0, T]$, is absolutely continuous with respect to the Lebesgue measure, with a density residing in $L^1(\mathbb{R}^{2N}) \cap L^\infty(\mathbb{R}^{2N})$.*

Proof. For all $\phi \in \mathfrak{C}_0^2$, we have

$$\int_{\mathbb{R}^{2N}} \int \phi(x, v)\, d\mu_t^{(n)}(x, v) = \int_{\mathbb{R}^{2N}} \int \phi(x, v)\, f^{\delta_n}(t, x, v)\, dv\, dx$$

$$\to \int_{\mathbb{R}^{2N}} \int \phi(x, v)\, d\mu_t(x, v).$$

We observe that

$$\left| \int_{\mathbb{R}^{2N}} \int \phi(x, v)\, d\mu_t^{(n)}(x, v) \right| \leqslant \|f_0\|_\infty\, e^{N\beta t} \int_{\mathbb{R}^{2N}} \int |\phi(x, v)|\, dv\, dx$$

$$= \|f_0\|_\infty\, e^{N\beta t}\, \|\phi\|_1. \tag{3.23}$$

Hence, we obtain

$$\left| \int_{\mathbb{R}^{2N}} \int \phi(x, v)\, d\mu_t(x, v) \right| \leqslant \|f_0\|_\infty\, e^{N\beta t}\, \|\phi\|_1, \qquad \phi \in \mathfrak{C}_0^2,\ t \in [0, T].$$

For each t, then, the functional $\phi \mapsto \int_{\mathbb{R}^{2N}} \int \phi(x, v)\, d\mu_t(x, v)$ is a positive, linear, and bounded functional on the dense subset $\mathfrak{C}_0^2 \in L^1(\mathbb{R}^{2N})$. It has a unique continuation as a positive and bounded functional on $L^1(\mathbb{R}^{2N})$ with norm dominated by $\|f_0\|_\infty\, e^{N\beta t}$. There is a $f(t, \cdot, \cdot) \in L^\infty(\mathbb{R}^{2N})$ with

$$\|f(t, \cdot, \cdot)\|_\infty \leqslant \|f_0\|_\infty\, e^{N\beta t},$$

such that

$$\int_{\mathbb{R}^{2N}} \int \phi(x, v)\, d\mu_t(x, v) = \int_{\mathbb{R}^{2N}} \int \phi(x, v)\, f(t, x, v)\, dv\, dx \tag{3.24}$$

for all $\phi \in \mathfrak{C}_0^2$. It is clear that $f(t, \cdot, \cdot) \geqslant 0$ a.e.

We now proceed to show that $f(t, \cdot, \cdot) \in L^1(\mathbb{R}^{2N})$ also. Toward this end, we exploit the tightness property of the sequence $\{\mu_t^{(n)}: t \in [0, T]\}$, as described by Lemma III.1. The following deliberations are reminiscent of arguments in the classical Vitali convergence theorem: Let $\varepsilon > 0$ be given and select an $R > 0$ such that

$$B - \varepsilon \leqslant \int_{\{(x, v): |x|^2 + |v|^2 \leqslant R^2\}} \int f^{\delta_n}(t, x, v)\, dv\, dx, \qquad B := \|f_0\|_1 \tag{3.25}$$

for δ_n, $\delta_n \downarrow 0$. Suppose $\phi^\varepsilon(x, v)$ is a function, residing in $\mathfrak{C}_0^2$ such that $0 \leqslant \phi^\varepsilon(x, v) \leqslant 1$ and

$$\phi^\varepsilon(x, v) := \begin{cases} 1, & |x|^2 + |v|^2 \leqslant R^2 \\ 0, & |x|^2 + |v|^2 > (R + \varepsilon)^2 \end{cases} \tag{3.26}$$

Then we observe that for all δ_n, $\delta_n \downarrow 0$,

$$B-\varepsilon \leqslant \int_{\{(x,v):|x|^2+|v|^2 \leqslant R^2\}} \int f^{\delta_n}(t,x,v)\,dv\,dx$$
$$\leqslant \int_{\mathbb{R}^{2N}} \int \phi^{\varepsilon}(x,v)\, f^{\delta_n}(t,x,v)\,dv\,dx \leqslant B. \tag{3.27}$$

The weak convergence of the measures $\mu_t^{(n)}$ implies that

$$B-\varepsilon \leqslant \int_{\mathbb{R}^{2N}} \int \phi^{\varepsilon}(x,v)\, f(t,x,v)\,dv\,dx \leqslant B, \qquad t \in [0,T] \tag{3.28}$$

for all such ϕ^{ε}. The summability of $f(t,\cdot,\cdot)$ follows and our limiting f is actually the density of μ_t. The weak continuity of $f(t,\cdot,\cdot)$, in the sense of Definition II.3, follows from that of μ_t. This completes the proof.

The conclusions of Lemmas III.1–III.4 yield the proof of Theorem 1.

IV. Existence of Weak Solutions for Plasma Physical Problems

In this section, we show that the density $f(t,\cdot,\cdot)$ of the limiting measure μ_t, defined on Borel sets of $\mathbb{R}^{2N}$, is a weak solution of (1.1)–(1.4) in the sense of Definition II.4. Our existence theorem is the following:

THEOREM 2. *Suppose the assumptions on f_0, as stipulated in Theorem* 1, *hold. Then the densities $f(t,\cdot,\cdot)$ of the measures μ_t, $t \in [0,T]$, constitute a weak solution of* (1.1)–(1.4).

Proof. We must verify items (i) and (ii) of Definition II.4. Certainly, item (i) is clear according to the concluding paragraphs of Lemma III.4. For showing item (ii), we know that indeed $f^{\delta_n}(t,\cdot,\cdot)$ is a weak solution of the mollified problem (2.11)–(2.14) with mollification parameter δ_n, since it is a classical solution for $t>0$. For arbitrary $\phi \in \mathfrak{C}_0^2([0,T]\times\mathbb{R}^{2N})$, then, we have

$$\int_0^T \left(\int_{\mathbb{R}^{2N}} \int f^{\delta_n}(t,x,v) \left[\frac{\partial \phi}{\partial t}(t,x,v) + v\cdot\nabla_x \phi(t,x,v) \right.\right.$$
$$\left.\left. +\ (\mathbf{E}^{\delta_n}(t,x) - \beta v)\cdot\nabla_v \phi(t,x,v) - \sigma \Delta_v \phi(t,x,v) \right] dv\,dx \right) dt$$
$$+ \int_{\mathbb{R}^{2N}} \int \phi(0,x,v)\, f_0(x,v)\,dv\,dx = 0. \tag{4.1}$$

It behooves us to show that all terms depending on n, or on δ_n, converge as $n \to \infty$ to the corresponding terms in (2.4).

We first consider

$$\Upsilon(t, x, v) := \frac{\partial \phi}{\partial t}(t, x, v) + v \cdot \nabla_x \phi(t, x, v) - \beta v \cdot \nabla_v \phi(t, x, v) - \sigma \Delta_v \phi(t, x, v). \tag{4.2}$$

From the properties of ϕ, we have the existence of $R > 0$ such that $\Upsilon(t, x, v) = 0$ if $|x|^2 + |v|^2 > R^2$, $t \in [0, T]$. Clearly $\Upsilon(t, x, v) \in \mathfrak{C}_0^0([0, T] \times \mathbb{R}^{2N})$, and $\Upsilon(t, \cdot, \cdot) \in \mathfrak{C}_0^0$. The fact that

$$u_t^{(n)} \xrightarrow{w} \mu_t$$

implies by definition

$$\int_{\mathbb{R}^{2N}} \int f^{\delta_n}(t, x, v)\, \Upsilon(t, x, v)\, dv\, dx \to \int_{\mathbb{R}^{2N}} \int f(t, x, v)\, \Upsilon(t, x, v)\, dv\, dx \tag{4.3}$$

for each $t \in [0, T]$. Because

$$\left| \int_{\mathbb{R}^{2N}} \int f^{\delta_n}(t, x, v)\, \Upsilon(t, x, v)\, dv\, dx \right| \leqslant \|f_0\|_1 \sup_{t \in [0, T]} \|\Upsilon(t, \cdot, \cdot)\|_\infty, \tag{4.4}$$

we can exploit the Lebesgue dominated convergence theorem to assert that

$$\int_0^T \left(\int_{\mathbb{R}^{2N}} \int f^{\delta_n}(t, x, v)\, \Upsilon(t, x, v)\, dv\, dx \right) dt \to \int_0^T \left(\int_{\mathbb{R}^{2N}} \int f(t, x, v)\, \Upsilon(t, x, v)\, dv\, dx \right) dt. \tag{4.5}$$

We next must investigate the convergence of

$$\int_0^T \int_{\mathbb{R}^{2N}} \int \mathbf{E}^{\delta_n}(t, x) \cdot \nabla_v \phi(t, x, v)\, f^{\delta_n}(t, x, v)\, dv\, dx\, dt.$$

Toward this end, we consider the sequence of functions

$$\Xi_n(t) := \int_{\mathbb{R}^{2N}} \int \mathbf{E}^{\delta_n}(t, x) \cdot \nabla_v \phi(t, x, v)\, f^{\delta_n}(t, x, v)\, dv\, dx, \tag{4.6}$$

and note that $\Xi_n(t)$ can be decomposed as

$$\begin{aligned}
&\Xi_n(t)\\
&= \int_{\mathbb{R}^{2N}}\int \left(\int_{\mathbb{R}^N} k_n(x-y)\cdot\nabla_v\phi(t,x,v)\,\rho^{\delta_n}(t,y)\,dy\right) f^{\delta_n}(t,x,v)\,dv\,dx\\
&= \int_{\mathbb{R}^{2N}}\int \left(\int_{\mathbb{R}^{2N}}\int k_n(x-y)\nabla_v\phi(t,x,v)\,f^{\delta_n}(t,y,w)\,dw\,dy\right) f^{\delta_n}(t,x,v)\,dv\,dx\\
&\quad -\int_{\mathbb{R}^{2N}}\int \left(\int_{\mathbb{R}^{2N}}\int k_n(x-y)\cdot\nabla_v\phi(t,x,v)\,N_0(y,w)\,dw\,dy\right) f^{\delta_n}(t,x,v)\,dv\,dx\\
&= \int_{\mathbb{R}^{2N}}\int [f^{\delta_n}(t,y,w)-N_0(y,w)]\int_{\mathbb{R}^{2N}}\int k_n(x-y)\\
&\quad \cdot\nabla_v\phi(t,x,v)\,f^{\delta_n}(t,x,v)\,dv\,dx\,dw\,dy. \qquad (4.7)
\end{aligned}$$

In order to examine the convergence of $\Xi_n(t)$ more easily, we let $\zeta_n(t, y)$ denote the inner integral, namely,

$$\zeta_n(t, y)=\int_{\mathbb{R}^{2N}}\int k_n(x-y)\cdot\nabla_v\phi(t, x, v)\,f^{\delta_n}(t, x, v)\,dv\,dx. \qquad (4.8)$$

We first observe that $\{\zeta_n(t, y)\colon n\in\mathbb{N}\}$ is uniformly bounded on $[0, T]\times\mathbb{R}^N$. This is due to the fact that $\nabla_v\phi(t, x, v)=0$ for all $t\in[0, T]$ and all x and v residing outside a ball of some radius R. We estimate

$$|\zeta_n(t, y)|\leqslant \sup_{t\in[0,T]}\|\nabla_v\phi(t,\cdot,\cdot)\|_\infty\int_{\operatorname{supp}\nabla_v\phi}\int |k_n(x-y)|\,f^{\delta_n}(t, x, v)\,dv\,dx, \qquad (4.9)$$

where $\operatorname{supp}\nabla_v\phi$ denotes the support of $\nabla_v\phi$ in $\mathbb{R}^{2N}$. Hence, $\zeta_n(t, y)$ is uniformly bounded in n, t, and y for fixed $\phi\in\mathfrak{C}_0^2([0, T]\times\mathbb{R}^{2N})$ because of Lemma III.2.

We can actually say more: $\zeta_n(t, y)$ is uniformly convergent on $[0, T]\times\mathbb{R}^N$. Let $\varepsilon>0$ be given. We observe the following:

(A) There is an $R_d>0$ such that

$$\left|\int_{\mathbb{R}^N}\left(\int_{|x-y|<R_d} k_n(x-y)\cdot\nabla_v\phi(t, x, v)\,f^{\delta_n}(t, x, v)\,dx\right)dv\right|\leqslant\varepsilon/3 \qquad (4.10)$$

for all $t\in[0, T]$, $y\in\mathbb{R}^N$, and all n. To see this, note that

$$\begin{aligned}
&\left|\int_{\mathbb{R}^N}\left(\int_{\{x:|y-x|\leqslant d\}} k_n(x-y)\,\nabla_v\phi(t, x, v)\,f^{\delta_n}(t, x, v)\,dx\right)dv\right|\\
&\qquad\leqslant \sup_{[0,T]}\|\nabla_v\phi(t,\cdot,\cdot)\|_\infty\int_{\{x:|x-y|\leqslant d\}}|k_n(x-y)|\,\tilde\rho^{\delta_n}(t, x)\,dx, \qquad (4.11)
\end{aligned}$$

where $\tilde{\rho}^{\delta_n}(t, x)$ is given by

$$\tilde{\rho}^{\delta_n}(t, x) := \int_{v-\operatorname{supp} \nabla_v \phi} f^{\delta_n}(t, x, v)\, dv \tag{4.12}$$

with $v - \operatorname{supp} \nabla_v \phi$ denoting the velocity support of $\nabla_v \phi$. The expression (4.11) can be bounded by

$$\begin{aligned}
&\sup_{[0,T]} \|\nabla_v \phi(t, \cdot, \cdot)\|_\infty \int_{\{x : |x-y| \leqslant d\}} |k_n(x-y)|\, \tilde{\rho}^{\delta_n}(t, x)\, dx \\
&\quad \leqslant \sup_{[0,T]} \|\nabla_v \phi(t, \cdot, \cdot)\|_\infty \sup_{[0,T]} \|\tilde{\rho}^{\delta_n}(t, \cdot)\|_\infty \int_{\{\xi : |\xi| \leqslant d\}} |k_n(\xi)|\, d\xi. \qquad (4.13)
\end{aligned}$$

The conclusion of observation (A) will be effected if we can show that $|x|^{N-1}\, |k_n(x)|$ is uniformly bounded. Toward this end, we note that

$$\begin{aligned}
|x|^{N-1}\, |k_n(x)| &\leqslant \int_{\mathbb{R}^N} |x|^{N-1}\, |x-y|^{-(N-1)} \left| \omega\left(\frac{y}{\delta}\right) \right| \delta^{-N}\, dy \\
&\leqslant \int_{\mathbb{R}^N} [\,|x-y| + |y|\,]^{N-1}\, |x-y|^{-(N-1)}\, \omega\left(\frac{y}{\delta}\right) \delta^{-N}\, dy \\
&= \sum_{j=0}^{N-1} \binom{N-1}{j} |x-y|^{-N+j+1}\, |y|^{N-1-j}\, \delta^{-N} \left| \omega\left(\frac{y}{\delta}\right) \right| dy. \qquad (4.14)
\end{aligned}$$

We need to estimate, then, convolution integrals of the form,

$$\int_{\mathbb{R}^N} |x-y|^{-(N-1)+j}\, |y|^{N-1-j}\, \delta^{-N} \left| \omega\left(\frac{y}{\delta}\right) \right| dy, \qquad \text{with } j < N-1.$$

(For $j = N-1$, we have merely $\|\omega\|_1$). The general case will follow from a lemma by E. Horst [7, p. 233]. To apply this lemma, we let $\alpha = N - (j+1)$, $p \in (1, \infty]$, $q \in [1, \infty)$, with $p > N/(j+1) > q$. We have that

$$\begin{aligned}
&\int_{\mathbb{R}^N} |x-y|^{-(N-1)+j}\, |y|^{N-(j+1)} \delta^{-N} \left| \omega\left(\frac{y}{\delta}\right) \right| dy \\
&\quad \leqslant C(N, j, p, q) \left[\int_{\mathbb{R}^N} |y|^{p(N-j-1)} \left| \omega^p \left(\frac{y}{\delta}\right) \right| \delta^{-Np}\, dy \right]^{\lambda/p} \\
&\qquad \cdot \left[\int_{\mathbb{R}^N} |y|^{q(N-j-1)} \left| \omega^q \left(\frac{y}{\delta}\right) \right| \delta^{-Nq}\, dy \right]^{(1-\lambda)/q}, \qquad (4.15)
\end{aligned}$$

where

$$\lambda = \frac{(N-j-1)/N - 1 + 1/q}{1/q - 1/p}, \quad 1 - \lambda = \frac{1-(N-j-1)/N - 1/p}{1/q - 1/p},$$

and $C(N, j, p, q)$ is a universal constant depending at most on the indicated parameters. An elementary but tedious calculation shows that

$$\begin{aligned} \int_{\mathbb{R}^N} |x-y|^{-(N-1)+j} \, |y|^{N-(j+1)} \, \delta^{-N} \left| \omega\left(\frac{y}{\delta}\right) \right| dy \\ \leqslant C(N, j, p, q) \left[\int_{\mathbb{R}^N} |y|^{p(N-j-1)} | \, \omega^p(y)| \, dy \right]^{\lambda/p} \\ \cdot \left[\int_{\mathbb{R}^N} |y|^{q(N-j-1)} \, |\omega^q(y)| \, dy \right]^{(1-\lambda)/q}, \end{aligned} \tag{4.16}$$

and observation (A) results owing to the fact that ω is a testing function of rapid descent. A similar analysis shows observation (A) true for the limiting case when $k_n(x-y)$ is replaced by $(x-y)\,|x-y|^{-N}$.

(B) Let us fix y for the moment. Then we note that the mappings

$$x \mapsto k_n(x-y)$$

are continuous in $\{x : |x-y| \geqslant d\}$, and an easy manipulation of the convolution integral defining $k_n(x)$ in (2.7) indicates that $|k_n(x-y) + \lambda\omega_N^{-1}(x-y)\,|x-y|^{-N}| \to 0$ uniformly for all (x, y) with $|x-y| \geqslant d$. For each t, then, the functions

$$(x, v) \mapsto k_n(x-y) \cdot \nabla_v \phi(t, x, v)$$

are of class $\mathfrak{C}_0^0$ on the domain $\{x : |x-y| \geqslant d\} \times \mathbb{R}^N$, and these converge uniformly to

$$-\lambda\omega_N^{-1}(x-y)\,|x-y|^{-N} \cdot \nabla_v \phi(t, x, v).$$

This convergence is uniform with respect to t and y also, since for given $\eta > 0$, the corresponding $\eta = M_0$ depends only on η and the selection of $\phi \in \mathfrak{C}_0^2([0, T] \times \mathbb{R}^N \times \mathbb{R}^N)$.

Since $\mu_t^{(n)} \xrightarrow{w} \mu_t$, we have for any $\phi \in \mathfrak{C}_0^2$,

$$\int_{\mathbb{R}^{2N}} \phi(x, v) \, d\mu_t^{(n)} \to \int_{\mathbb{R}^{2N}} \phi(x, v) \, d\mu_t$$

uniformly in t, because of Lemma III.3. An easy density argument shows this true for any $\phi \in \mathfrak{C}_0^0$ and when $\mathbb{R}^{2N}$ is replaced by $\{x: |x-y| \geqslant d\} \times \mathbb{R}^N$. Therefore, there is an $M_0(\varepsilon)$ independent of y and t such that

$$\left| \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} [k_n(x-y) f^{\delta_n}(t, x, v) + \lambda \omega_N^{-1}(x-y) |x-y|^{-N} f(t, x, v)] \right.$$
$$\left. \cdot \nabla_v \phi(t, x, v) \, dv \, dx \right| < \varepsilon$$

for all $n \geqslant M_0(\varepsilon)$.

Since $\zeta_n(t, y)$ is continuous in t and y for every n, then so is $\zeta(t, y)$, where

$$\zeta(t, y) := -\lambda \omega_N^{-1} \int_{\mathbb{R}^{2N}} \int (x-y) |x-y|^{-N} \cdot \nabla_v \phi(t, x, v) f(t, x, v) \, dv \, dx.$$

Therefore, since $\|\zeta_n(t, \cdot) - \zeta(t, \cdot)\|_\infty \to 0$ as $n \to \infty$ (or $\delta_n \to 0$) uniformly in t, we obtain the result that

$$\int_{\mathbb{R}^{2N}} \int [f^{\delta_n}(t, y, v) - N_0(y, v)] \, \zeta_n(t, y) \, dv \, dy$$
$$\to \int_{\mathbb{R}^{2N}} \int [f(t, y, v) - N_0(y, v)] \, \zeta(t, y) \, dv \, dy$$

uniformly in $t \in [0, T]$. Thus

$$\int_0^T \int_{\mathbb{R}^{2N}} \int [f^{\delta_n}(t, y, v) - N_0(y, v)] \, \zeta_n(t, y) \, dv \, dy \, dt$$
$$\to \int_0^T \int_{\mathbb{R}^{2N}} \int [f(t, y, v) - N_0(y, v)] \, \zeta(t, y) \, dv \, dy \, dt,$$

as $\delta_n \to 0$. This completes the proof of Theorem 2.

V. Remarks on Stellar Dynamical Models

In the preceding section, we have seen that for plasma physical models, we have the existence of global weak solutions satisfying (1.1) in the sense of Definition II.4. This is fundamentally due to the fact that the kinetic energies possess a bound independent of the mollification parameter δ, but possibly depending on the time interval selected. This will not be the case for the stellar dynamical setting, since the potential energies here constitute a *negative definite quadratic form*, owing to the presence of positive λ.

We proceed to supply an example which shows that only local existence of weak solutions for $N \geqslant 4$ can only be guaranteed for stellar dynamical-type cases. For our initial data we select

$$f_0(x, v) = \gamma \pi^{-N} \exp(-|x|^2 - |v|^2). \tag{5.1}$$

The theory developed in [13] assures us of the local existence of classical solutions to (1.1)–(1.4). With the data in (5.1), we show that there is a time T when both classical solutions and weak solutions fail to exist. As we shall see, the distribution evolves into a measure which will not be absolutely continuous with respect to the Lebesgue measure, and hence Definition II.4 (i) is violated. At this juncture, we remind the reader that $E(t) := \mathfrak{K}\mathfrak{E}(t) + \mathfrak{P}\mathfrak{E}(t)$, with $h(t)$ given by (2.33).

Toward proving these remarks, we first note that for $f_0(x, v)$ in (5.1)

$$\frac{dh}{dt}(0) = 2 \int_{\mathbb{R}^{2N}} \int (x, v) f_0(x, v)\, dv\, dx = \frac{2\gamma}{\pi^N} \int_{\mathbb{R}^{2N}} x \exp(-|x|^2)\, dx$$
$$\cdot \int_{\mathbb{R}^N} v \exp(-|v|^2)\, dx = 0. \tag{5.2}$$

The parameter γ is at our disposal to choose. Let us now select T at the outset. Then, from (2.36), we note that for $0 \leqslant t \leqslant T$,

$$\begin{aligned}
\frac{dh}{dt} &\leqslant e^{-\beta t} \frac{dh}{dt}(0) + 2 \int_0^t e^{-\beta(t-s)} [2N\sigma \|f_0\|_1 s + E(0)]\, ds \\
&\quad + (N-4) \int_0^t e^{-\beta(t-s)} \mathfrak{P}\mathfrak{E}(s)\, ds \\
&= e^{-\beta t} \frac{dh}{dt}(0) + 4N\sigma \|f_0\|_1 \left[\frac{\beta t - 1 + e^{-\beta t}}{\beta^2} \right] \\
&\quad + \frac{2}{\beta}(1 - e^{-\beta t})(\mathfrak{K}\mathfrak{E}(0) + \mathfrak{P}\mathfrak{E}(0)) \\
&\quad + (N-4) \int_0^t e^{-\beta(t-s)} \mathfrak{P}\mathfrak{E}(s)\, ds.
\end{aligned} \tag{5.3}$$

Integrating once more, we obtain

$$
\begin{aligned}
h(t) \leqslant h(0) &+ 4N\sigma \, \|f_0\|_1 \left[\frac{\beta^2 t^2 - 2\beta t + 2 - 2e^{-\beta t}}{2\beta^3} \right] \\
&+ \frac{2(\beta t - 1 + e^{-\beta t})}{\beta^2} (\mathfrak{K}\mathfrak{E}(0) + \mathfrak{P}\mathfrak{E}(0)) \\
&+ (N-4) \int_0^t \int_0^\tau e^{-\beta(\tau - s)} \mathfrak{P}\mathfrak{E}(s) \, ds \, d\tau \\
\leqslant h(0) &+ 4N\sigma \, \|f_0\|_1 \left[\frac{\beta^2 t^2 - 2\beta t + 2 - 2e^{-\beta t}}{2\beta^3} \right] \\
&+ \frac{2}{\beta^2} (\beta t - 1 + e^{-\beta t})(\mathfrak{K}\mathfrak{E}(0) + \mathfrak{P}\mathfrak{E}(0)). \qquad (5.4)
\end{aligned}
$$

With the data so specified, we have for arbitrary $t \in [0, T]$,

$$
\begin{aligned}
h(t) \leqslant \frac{\gamma N}{2} &+ 2N\sigma\gamma \left[\frac{\beta^2 t^2 - 2\beta t + 2 - 2e^{-\beta t}}{\beta^3} \right] \\
&+ \frac{\gamma N(\beta t - 1 + e^{-\beta t})}{\beta^2} - \frac{2}{\beta^2} (\beta t - 1 + e^{-\beta t}) \frac{\gamma^2 \lambda}{\omega_N (N-2) \pi^N} \\
&\times \int_{\mathbb{R}^{2N}} \int |x - y|^{-(N-2)} \exp(-|x|^2 - |y|^2) \, dy \, dx \qquad (5.5)
\end{aligned}
$$

with $\lambda > 0$. Observe that this is a pointwise estimate for each $t \in [0, T]$. At time $t = T$, we have then

$$
\begin{aligned}
h(T) \leqslant \frac{\gamma N}{2} &+ 2N\sigma\gamma \left[\frac{\beta^2 T^2 - 2\beta T + 2 - 2e^{-\beta T}}{\beta^3} \right] \\
&+ \gamma N \frac{(\beta T - 1 + e^{-\beta T})}{\beta^2} - \frac{2(\beta T - 1 + e^{-\beta T})}{\beta^2} \frac{\lambda \gamma^2}{\omega_N (N-2) \pi^N} \\
&\times \int_{\mathbb{R}^{2N}} \int |x - y|^{-(N-2)} \exp(-|x|^2 - |y|^2) \, dy \, dx. \qquad (5.6)
\end{aligned}
$$

Now choose γ so large and positive that the right-hand side is negative. This is always possible for arbitrary, but fixed T, since the potential energy at $t = 0$ is quadratic in γ. With such a value of γ, depending on T, we see that h will become negative for a finite terminal time T. The distribution $f(t, \cdot, \cdot)$ has then failed to exist as an $L^1(\mathbb{R}^{2N})$ function, and has possibly evolved into a Borel measure which is not absolutely continuous with respect to the Lebesgue measure. For any T, there is a multiple of a

Maxwellian-type initial distribution for which the *Liouville–Newton–Fokker–Planck dynamics* will lead to gravitational collapse.

But does this happens for $N=3$? In other words, can we show that the kinetic energies remain uniformly bounded in the mollification parameter on arbitrary time intervals? An argument by E. Horst [8], adapted to our setting, will show this true for arbitrary data for $N=3$, but for data suitably "small" for $N \geqslant 4$. We take as our mollified potential

$$\mu_\delta(x) = \frac{-\lambda}{(N-2)\,\omega_N}\,(|x|^2+\delta)^{-(N-2)/2}. \tag{5.7}$$

The analysis by E. Horst employs a Sobolev estimate to show that the potential energy for each value of δ is dominated by a multiple of the square root of the associated kinetic energy. An easy generalization of Horst's arguments will indicate that the family of kinetic energies, $N=3$, possesses the following bound for each t:

$$\begin{aligned} 0 \leqslant 2(\mathfrak{KE}^\delta(t))^{1/2} &\leqslant \tilde{C}_3\, \|f_0\|_\infty^{1/3}\, \|f_0\|_1^{1/6} \exp(\beta t) \\ &+ \sqrt{\tilde{C}_3^2\, \|f_0\|_\infty^{2/3}\, \|f_0\|_1^{7/3} \exp(2\beta t) + 4(\mathfrak{KE}(0) + \mathfrak{PE}^\delta(0)) + 24\sigma\, \|f_0\|_1\, t}, \end{aligned} \tag{5.8}$$

where

$$\tilde{C}_3 := 27\sqrt{2}\,\sqrt[5]{6}\left(1+\left(\frac{4}{3}\pi\right)^{2/9}\right)^3 \lambda \Big/ 2\pi \tag{5.9}$$

and $\{\mathfrak{PE}^\delta(0), \delta \geqslant 0\}$ is uniformly bounded in δ as seen by the form of the mollified potential in (5.7). For $N=4$, on the other hand, the Horst estimates yield the following

$$\begin{aligned} &\mathfrak{KE}^\delta(t) \\ &\quad \leqslant (8\sigma\, \|f_0\|_1\, t + \mathfrak{PE}(0) + \mathfrak{KE}(0))/(1 - \tilde{C}_4\, \|f_0\|_\infty^{1/2}\, \|f_0\|_1^{1/2} \exp 2\beta t) \end{aligned} \tag{5.10}$$

with

$$\tilde{C}_4 := \frac{4^{19/3}\lambda}{9\omega_4}\left(1+\left(\frac{\omega_4}{4}\right)^{1/2}\right)^2. \tag{5.11}$$

Expression (5.10) imposes a condition of smallness on the initial data, once a time interval has been selected, in order to guarantee a uniform bound on the family of kinetic energies. The analysis for $N \geqslant 5$ is similar. We have, then, the following fundamental existence result for stellar dynamical models.

THEOREM 3 (Existence for Stellar Dynamical Models). *A necessary and sufficient condition for the existence of global weak solutions to the Liouville–Newton–Poisson–Fokker–Planck system* (1.1)–(1.4), *with* $f_0(x, v)$ *satisfying the conditions in* (2.8)–(2.10) *and* $N_0(x, v) \equiv 0$, *is that the spatial or momentum dimension be less than or equal to* 3.

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